Representations of the discrete inhomogeneous Lorentz group and Dirac wave equation on the lattice

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Abstract.   We propose the fundamental and two-dimensional representation of the Lorentz groups on a (3+1)-dimensional hypercubic lattice, from which representations of higher dimensions can be constructed. For the unitary representation of the discrete translation group we use the kernel of the Fourier transform. From the Dirac representation of the Lorentz group (including reflections) we derive in a natural way the wave equation on the lattice for spin-1/2 particles. Finally, the induced representation of the discrete inhomogeneous Lorentz group is constructed by standard methods and its connection with the continuous case is discussed.

1. Introduction

The hypothesis of a discrete space and time has attracted the attention of physicists for different reasons [1, 3]:

 (i) As a mathematical tool in order to remove the infinities in quantum field theories with the help of a cut-off in momentum space or, equivalently, a lattice for spacetime coordinates [7]. This model is similar to solid state physics where quantum fields are defined on grid points of a periodic crystal.
 (ii) As a more realistic interpretation of lattice gauge theories, in which the spacetime variables are constrained to discrete values due to some underlying structure resulting out of the relations among fundamental processes, as Penrose, Finkelstein and Weizsaecker have proposed [3].

In this paper we address ourselves to an important problem of symmetries in lattice theories. In particular, we study the consequences of restricting the continuous spacetime variables to a discrete Minkowski space for the translations, rotations and Lorentz transformations. We discuss the possibility of maintaining representation theory for these groups on the lattice.

Our paper is based on the standard theory of induced representations of the Poincaré group restricted to discrete variables [13]. In momentum space there are three ways to construct induced representations: Mackey, Wigner and covariant states [10], based on the existence of a closed subgroup of a Lie group. In our case the closed subgroup is the cubic group with respect to Lorentz transformations on the lattice. We have the advantage that the representations of the rotation group in two and three dimensions remain irreducible when restricted to the cubic group. Therefore, all the arguments for the discrete case can be taken unchanged from the

continuous case. So we introduce the Poincaré symmetry on the lattice that was considered broken by many authors [12].

In a preliminary version [15] of this work we stressed the connection between Klein–Gordon, Dirac and Proca equations in discrete/continuous momentum and discrete space via a Fourier transform.

In this paper we emphasize representation theory of the discrete translation, Lorentz and Poincaré group in such a way that the wave equation for spin-1/2 particles on the lattice emerges in a natural way from the Dirac representation of the Lorentz group.

In section 2 we describe an algorithm to construct all integral transformations of the complete Lorentz group based on the generators of some Coxeter group and calculate the two-dimensional (2D) representations of this group that can be generalized to higher dimensional irreducible representations.

In section 3 we review two unitary irreducible representations of the discrete translation group and the cyclic group and use these representations as the kernel of two Fourier transforms on the lattice that have become very helpful throughout the literature.

In section 4 we review the Dirac representations of the Lorentz group including space inversion and construct the Dirac wave equation in momentum space as projection operators that reduce the covariant states of the representation to the irreducible components.

In section 5 we apply the Fourier transforms of section 3 to the Dirac equation in momentum space and obtain a difference equation for the Dirac and Klein–Gordon fields on the lattice.

In section 6 we construct the induced representation of the Poincaré group on the lattice using the Mackey–Wigner approach and discuss the irreducibility and orbit conditions of this representation.

2. Fundamental and spin representation of the Lorentz group on the lattice

An integral Lorentz transformation belongs to $GL(4, Z)$ and leaves invariant the bilinear form

$$d^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2. \tag{1}$$

According to Coxeter [1, p 47] integral Lorentz transformations (including reflections) are obtained by combining the operations of permuting the spatial coordinates x_1, x_2, x_3 and changing the signs of any of the coordinates x_0, x_1, x_2, x_3 together with adding $x_0 - x_1 - x_2 - x_3$ to each of the four coordinates of a point.

These operations can be described geometrically by Weyl reflections on the planes perpendicular to the vectors

$$\alpha_1 = e_1 - e_2 \qquad \alpha_2 = e_2 - e_3 \qquad \alpha_3 = e_3 \qquad \alpha_4 = -(e_0 + e_1 + e_2 + e_3)$$

where $\{e_0, e_1, e_2, e_3,\}$ is an orthonormal basis.

In matrix form these reflections are

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad S_4 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}.$$

These reflections generate a Coxeter group, the Dynkin diagram of which is depicted in figure 1 [8].
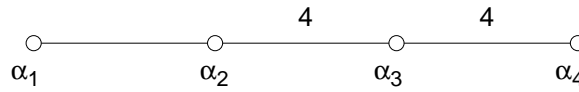
Figure 1. Dynkin diagram for the Coxeter group which generates all integral Lorentz transformations.

Kac [2] has proved that S_1, S_2, S_3, S_4 generate all the integral Lorentz transformations that keep invariant the upper half of the light cone. Note that S_1, S_2, S_3 generate the full cubic or octahedral group.

The generator S_1 can be used to factorize any integral Lorentz transformation that belongs to the complete (orthochronous) Lorentz group. Let

$$L \equiv \begin{pmatrix} a & e & f & g \\ b & & & \\ c & & * & \\ d & & & \end{pmatrix}$$

be an integral matrix of determinant $\Delta = \pm 1$, satisfying

$$L^{\mathrm{t}} g L = g \qquad g = \mathrm{diag}(1, -1, -1, -1) \tag{2}$$

and also $a \geqslant 1$. From

$$a^2 - b^2 - c^2 - d^2 = 1$$

it follows that only one of b, c, d can be zero. Suppose $a > 1$, then we apply S_1, S_2, S_3 to L from the left until b, c, d become non-positive integers. To the resulting matrix we apply S_4, we get

$$L' = \begin{pmatrix} a' & e' & f' & g' \\ b' & & & \\ c' & & * & \\ d' & & & \end{pmatrix}$$

with $a' = 2a + b + c + d$. Obviously

$$(a + b + c + d)(a - b - c - d) = 1 - 2bc - 2bd - 2cd < 0$$

therefore, $a + b + c + d < 0$ or

$$2a + b + c + d = a' < a.$$

By iteration of the same algorithm we get

$$a > a' > a'' > \cdots > a^{(k)} \geqslant 1.$$

The last inequality is a consequence of the fact that L and S_4 belong to the complete Lorentz group. Following this process we get an integral matrix with $a^{(k)} = 1$ which is a combination of S_1, S_2, S_3, giving all 48 elements of the cubic group on the lattice.

Therefore, a general integral Lorentz transformation of the complete Lorentz group L can be decomposed as

$$L = P_1^{\eta} P_2^{\theta} P_3^{\iota} S_4 \ldots S_4 P_1^{\delta} P_2^{\varepsilon} P_3^{\zeta} S_4 \{ S_1^{\alpha} S_2^{\beta} S_3^{\gamma} \}_{\text{all permutations}} \tag{3}$$

where $P_1 = S_1 S_2 S_3 S_2 S_1$, $P_2 = S_2 S_3 S_2$, $P_3 = S_3$ are matrices which change the sign of b, c, d and $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \iota \ldots = 0, 1$.

In the continuous case the boosts characterize the quotient of the Lorentz group with respect to the subgroup of rotations. These continuous boosts take the vector $(1, 0, 0, 0)$,

stable under rotations, to any point of the unit hyperboloid in the forward light cone. In the discrete case $(1, 0, 0, 0)$ is stable under the full cubic group acting on space coordinates. The coset representatives of the integral Lorentz group with respect to the cubic subgroup are obtained by taking in equation (3) the quotient with respect to all integral elements of the cubic group denoted by $\{S_1^\alpha S_2^\beta S_3^\gamma\}_{\text{all permutations.}}$. These coset representatives allow us to construct an algorithm to obtain any vector of the unit hyperboloid in the forward light cone from the vector $(1, 0, 0, 0)$. Letting (a, b, c, d) be any vector with integral components satisfying $a^2 - b^2 - c^2 - d^2 = 1, (a \geqslant 1)$. First, we apply to this vector the parity operators P_1, P_2 or P_3 defined in equation (3), in such a way that the components b, c, d become non-positive numbers. To the resulting vector we apply the operator S_4 obtaining a new vector (a', b', c', d'), $a'^2 - b'^2 - c'^2 - d'^2 = 1$, with $a' = 2a + b + c + d < a$. Following the same procedure we obtain new integral vectors satisfying $a > a' > a'' > \cdots > a^{(k)} \geqslant 1$. In the last step we obtain the vector $(1, 0, 0, 0)$ as required. Taking the product of the generators used in this algorithm in inverse order we get the coset representative that takes the vector $(1, 0, 0, 0)$ to the vector (a, b, c, d) and we call this coset representative the integral boost.

A realization of the general element of the proper Lorentz group can be obtained through the Cayley transform

$$L = (1 - X)(1 + X)^{-1}$$

where X must satisfy $X^T g + g X = 0$. The general expression for X is

$$X = \begin{pmatrix} 0 & r & s & t \\ r & 0 & n & -p \\ s & -n & 0 & q \\ t & p & -q & 0 \end{pmatrix}$$

with n, p, q, r, s, t arbitrary real numbers (see [3] for the corresponding explicit expression of L)

A particular case of integral Lorentz transformations are integral Lorentz transformations without rotations. These can be obtained with the help of Cayley parameters [3] making $n = p = q = 0$ and m, r, s, t integers. We can have the following cases:

(i) $m^2 - r^2 - s^2 - t^2 = 1$

$$L = \begin{pmatrix} 2m^2 - 1 & 2mr & 2ms & 2mt \\ 2mr & 2r^2 + 1 & 2rs & 2rt \\ 2ms & 2rs & 2s^2 + 1 & 2st \\ 2mt & 2rt & 2st & 2t^2 + 1 \end{pmatrix} \tag{4}$$

(ii) $m^2 - r^2 - s^2 - t^2 = 2$

$$L = \begin{pmatrix} m^2 - 1 & mr & ms & mt \\ mr & r^2 + 1 & rs & rt \\ ms & rs & s^2 + 1 & st \\ mt & rt & st & t^2 + 1 \end{pmatrix} \tag{5}$$

(iii) $m^2 - r^2 - s^2 - t^2 = -1$

$$L = \begin{pmatrix} -2m^2 - 1 & 2mr & 2ms & 2mt \\ 2mr & -2r^2 + 1 & -2rs & -2rt \\ 2ms & -2rs & -2s^2 + 1 & -2st \\ 2mt & -2rt & -2st & -2t^2 + 1 \end{pmatrix} \tag{6}$$

(iv) $m^2 - r^2 - s^2 - t^2 = -2$

$$L = \begin{pmatrix} -m^2 - 1 & mr & ms & mt \\ mr & -r^2 + 1 & -rs & -rt \\ ms & -rs & -s^2 + 1 & -st \\ mt & -rt & -st & -t^2 + 1 \end{pmatrix}. \tag{7}$$

The solutions of the diophantine equations in the four cases are obtained by application of all the Coxeter reflections as given in equation (3) to the vector $(1, 0, 0, 0)$ in case (i), to the vector $(2, 1, 1, 0)$ in case (ii), to the vector $(0, 1, 0, 0)$ in case (iii) and to the vector $(0, 1, 1, 0)$ and $(1, 1, 1, 1)$ in case (iv). See Kac [2, p 70]. The four cases correspond to the integral Lorentz transformations given by Schild [1, p 42] restricted to the pure Lorentz transformations without rotations.

Equations (4)–(7) can also be considered as particular cases of integral boost, that take the vector $(1, 0, 0, 0)$ to the vector defined by the first column of each of the four matrices. In the continuous case Møller [4] has given a general boost that takes the vector $(1, 0, 0, 0)$ to any vector on the unit hyperboloid. If we want to have a general matrix that takes the vector $(1, 0, 0, 0)$ to any integral vector (M, R, S, T) on the unit hyperboloid this is achieved by the square root of the first matrix of equation (4)

$$\sqrt{L} = \begin{pmatrix} M & R & S & T \\ R & 1 + \dfrac{R^2}{1 + M} & \dfrac{RS}{1 + M} & \dfrac{RT}{1 + M} \\ S & \dfrac{RS}{1 + M} & 1 + \dfrac{S^2}{1 + M} & \dfrac{ST}{1 + M} \\ T & \dfrac{RT}{1 + M} & \dfrac{ST}{1 + M} & 1 + \dfrac{T^2}{1 + M} \end{pmatrix}. \tag{8}$$

This matrix is the restriction of the general Lorentz transformation given by Møller [4] to the values

$$M = \frac{p_0}{m_0 c} \qquad R = \frac{p_1}{m_0 c} \qquad S = \frac{p_2}{m_0 c} \qquad T = \frac{p_3}{m_0 c}$$

satisfying $M^2 - R^2 - S^2 - T^2 = 1$.

In position space the spacetime coordinates of the lattice x_μ are integer numbers. They transform under integral Lorentz transformations into integral coordinates. The same is true for the increments Δx_μ.

In momentum space the components of the four-momentum are not integer numbers but they can be constructed with the help of integral coordinates, namely

$$p_\mu = m_0 c \left(\frac{c \Delta t}{((c\Delta t)^2 - (\Delta \vec{x})^2)^{1/2}}, \frac{\Delta \vec{x}}{((c\Delta t)^2 - (\Delta \vec{x})^2)^{1/2}} \right). \tag{9}$$

If Δx_μ transform under integral Lorentz transformations as a four-vector, p_μ will transform also as a four-vector because the denominator in each component is Lorentz invariant.

In general, there is no constraint between the values Δx_μ. However, if we impose the condition $(c\Delta t)^2 - (\Delta \vec{x})^2 = 1$ then the four-momentum can be considered an integral four vector multiplied by $m_0 c$, $m_0 c(M, R, S, T)$, satisfying $(M^2 - R^2 - S^2 - T^2) = 1$ with $M = c\Delta t$, $R = \Delta x_1$, $S = \Delta x_2$, $T = \Delta x_3$.

Using the homomorphism between the groups *SO(3,1)* and *SL(2,C)* we obtain the representation of integral Lorentz transformations as 2D complex matrices. From knowledge of Cayley parameters [5] for a general element of the proper Lorentz group, we read off the matrix elements of the corresponding $\alpha \in \mathrm{SL}(2, C)$

$$\alpha = \frac{1}{\sqrt{\Delta}} \begin{pmatrix} m - t + \mathrm{i}(n - \lambda) & -p - r + \mathrm{i}(q - s) \\ p - r + \mathrm{i}(q + s) & m + t - \mathrm{i}(n + \lambda) \end{pmatrix} \tag{10}$$

with

$$\Delta = m^2 - r^2 - s^2 - t^2 + m^2 + p^2 + q^2 - \lambda^2 \qquad m\lambda = nt + ps + qr.$$

For instance we can calculate the 2D representation of the Coxeter reflections S_1, multiplied by the parity operator P (in order to get an element of the proper Lorentz group) identifying its matrix elements with the Lorentz matrix given in terms of Cayley parameters. Easy calculations give the unique solutions

$$\alpha(PS_1) = \pm\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1-i \\ 1-i & 0 \end{pmatrix} \tag{11}$$

$$\alpha(PS_2) = \pm\frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix} \tag{12}$$

$$\alpha(PS_3) = \pm\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{13}$$

$$\alpha(PS_4) = \pm\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1-i \\ -1-i & 2i \end{pmatrix}. \tag{14}$$

From these matrices one can prove the expressions for the representation of the algebra for the Coxeter reflections, namely

$$\alpha(S_i S_j) = \alpha(PS_i)\alpha(PS_j) \qquad i \neq j \qquad i, j = 1, 2, 3$$
$$\alpha(PS_i)\alpha(PS_i) = -1 \qquad i = 1, 2, 3$$
$$\alpha(PS_i)\alpha(PS_4) = \alpha(S_i S_4) \qquad i = 1, 2, 3. \tag{15}$$

The elements $\alpha(PS_i)$, $i = 1, 2, 3$ generate the so-called octahedral binary or double group [14] excluding the parity. Together with $\alpha(PS_4)$ they generate part of a binary version of the Coxeter group.

The integral Lorentz transformations without rotations as given in equations (4)–(7) have 2D representations taking $n = p = q = \lambda = 0$ in equation (10) and the choices

 (i) $m^2 - r^2 - s^2 - t^2 = 1$
 (ii) $m^2 - r^2 - s^2 - t^2 = 2$
(iii) $m^2 - r^2 - s^2 - t^2 = -1$
(iv) $m^2 - r^2 - s^2 - t^2 = -2$

in equations (4)–(7). In order to complete the picture we have to add the 2D representation of the matrix $\sqrt{L}$ given in equation (8) which turns out to be

$$\alpha(\sqrt{L}) \equiv \kappa = \frac{1}{\sqrt{2(M+1)}}\begin{pmatrix} M+1+T & R-iS \\ R+iS & M+1-T \end{pmatrix} = \kappa^+. \tag{16}$$

The (2×2) matrix representation of the discrete vector (M, R, S, T), $M^2 - R^2 - S^2 - T^2 = 1$ can be obtained from the momentum in the rest system $(1, 0, 0, 0)$ transformed by the matrix (16), namely

$$\kappa\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\kappa^+ = \begin{pmatrix} M+T & R-iS \\ R+iS & M-T \end{pmatrix}. \tag{17}$$

3. Representations of the translation group on the lattice and Fourier transforms

In the continuous case the unitary irreducible representations of the translation group are one-dimensional (1D) and are given by the function

$$D^k(x) = \exp(i2\pi xk) \qquad x \in R \tag{18}$$

where k is a continuous parameter that characterizes the representation.

If we restrict the translations to a discrete set of points $x = j\varepsilon$, $j \in Z$, the representation becomes

$$D^k(j) = \exp(i2\pi k j \varepsilon) \tag{19}$$

where k is still continuous. This representation satisfies orthogonality relations

$$\int_{-1/2\varepsilon}^{1/2\varepsilon} \overline{D}^k(j) D^k(j') \, \mathrm{d}k = \frac{1}{\varepsilon}\delta_{jj'} \tag{20}$$

and completeness relations

$$\sum_{j=-\infty}^{\infty} \overline{D}^k(j) D^{k'}(j) = \frac{1}{\varepsilon}\delta(k - k'). \tag{21}$$

The parameter of the discrete translation group k is defined in the fundamental domain $-(1/2\varepsilon) \leqslant k \leqslant (1/2\varepsilon)$.

From this representation we construct the *type I Fourier transform on the lattice*

$$\hat{F}(k) = \sum_{j=-\infty}^{\infty} \exp(2\pi i k j \varepsilon) F(j) \tag{22}$$

$$F(j) = \int_{-1/2\varepsilon}^{1/2\varepsilon} \exp(-2\pi i k j \varepsilon) \hat{F}(k) \, \mathrm{d}k. \tag{23}$$

If we generalize the discrete translation group to a (3+1)-dimensional cubic lattice

$$\Gamma : \{x_\mu = j_\mu \varepsilon, \ \mu = 0, 1, 2, 3, \ j_\mu \in Z\}$$

with scalar product

$$k \cdot x = k^\mu j_\mu \varepsilon = (k \cdot j)\varepsilon \tag{24}$$

then the representations become

$$D^k(j_\mu) = \exp 2\pi i (k \cdot j)\varepsilon. \tag{25}$$

These representations satisfy orthogonal relations, completeness relations and a Fourier transform analogous to the 1D case.

We introduce the reciprocal group of discrete translations on the reciprocal lattice Γ^R by

$$\Gamma^R : \left\{ b^R = b_\mu \frac{1}{\varepsilon}, b_\mu \in Z^4 \right\}.$$

The irreducible representations of the reciprocal group of discrete translations can be written

$$D^\xi(b^R) = \exp(2\pi i \xi \cdot b^R) \qquad -\varepsilon/2 \leqslant \xi_\mu \leqslant \varepsilon/2 \tag{26}$$

satisfying

$$\int_{-\varepsilon/2}^{\varepsilon/2} \overline{D}^\xi(b^R) D^\xi(b^{R'}) \, \mathrm{d}^4\xi = \varepsilon^4 \delta_{b^R, b^{R'}} \tag{27}$$

$$\sum_{b^R \in \Gamma^R}^{\infty} \overline{D}^\xi(b^R) D^{\xi'}(b^R) = \varepsilon^4 \delta(\xi - \xi'). \tag{28}$$

The representations of the translation group in the $(3+1)$-cubic lattice are invariant under the reciprocal group

$$D^k(j_\mu) = D^{k+b^R}(j_\mu) \tag{29}$$

therefore, the parameters of the representations k are restricted to the fundamental domain

$$-\frac{1}{2\varepsilon} < k_\mu \leqslant \frac{1}{2\varepsilon}.$$

The same property applies to the Fourier transform in $(3+1)$ dimensions

$$\hat{F}(k) = \hat{F}(k + b^R) \qquad b^R \in \Gamma^R. \tag{30}$$

Now we combine the irreducible representations of the discrete translations group in $(3+1)$ dimensions and the integral Lorentz transformations.

Given a periodic function on k-space

$$\hat{F}(k) = \hat{F}(k + b^R) \tag{31}$$

it can be written in terms of discrete waves

$$\hat{F}(k) = \sum_{j_\mu \in \Gamma} \exp(2\pi i (k \cdot j)\varepsilon) F(j_\mu). \tag{32}$$

From the action of the Lorentz group we have

$$U_\Lambda \hat{F}(k) = \hat{F}(\Lambda^{-1}k).$$

Then

$$U_\Lambda \hat{F}(k + b^R) = \hat{F}(\Lambda^{-1}k + \Lambda^{-1}b^R).$$

However, from the properties of the scalar product and using (32) we have

$$\exp(2\pi i (\Lambda^{-1}b^R \cdot j)\varepsilon) = \exp 2\pi i (b^R \cdot (\Lambda^{-1})^T j)\varepsilon = \exp 2\pi i (b_\mu^R (\Lambda^{-1})_\nu^{T\mu} j^\nu) = 1$$

for any integral Lorentz transformation. Finally

$$U_\Lambda \hat{F}(k + b^R) = \hat{F}(\Lambda^{-1}k) = U_\Lambda \hat{F}(k). \tag{33}$$

Therefore, the periodicity of some function with respect to the reciprocal group is conserved under integral Lorentz transformations.

Now we consider the 1D unitary irreducible representation of the cyclic group in 1D of order N

$$D^m(j) = \exp \frac{2\pi i}{N} mj \qquad m, j = 0, 1, \ldots, N-1 \tag{34}$$

where j represents the space variable and m is the label of the representation.

This representation satisfies periodic boundary conditions with respect to the j variable

$$D^m(j + N) = D^m(j)$$

and also with respect to the m label

$$D^{m+N}(j) = D^m(j)$$

and satisfies orthogonality relations

$$\frac{1}{N} \sum_{m=0}^{N-1} D^m(j) D^m(j') = \delta_{jj'} \tag{35}$$

and completeness relations

$$\frac{1}{N\varepsilon} \sum_{j=0}^{N-1} D^m(j) D^{m'}(j) = \frac{1}{\varepsilon} \delta_{mm'}. \tag{36}$$

From this representation we construct a *type II Fourier transform on the lattice*

$$\hat{F}(m) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \exp\left(i\frac{2\pi}{N}mj\right) F(j) \tag{37}$$

for any periodic function of discrete variables $F(j + N) = F(j)$. This is called, in literature, the finite Fourier transform.

The representation of the cyclic group in a 1D lattice and the corresponding Fourier transform can be generalized to the $(3 + 1)$-dimensional cubic lattice

$$D^m(j_\mu) = \exp i\frac{2\pi}{N}(m \cdot j) \tag{38}$$

with $(m \cdot j) = m^\mu j_\mu$ the bilinear form invariant under integral Lorentz transformations.

Because of the periodic boundary conditions the label of the representation of the cyclic group on the lattice is constrained to the fundamental domain, namely

$$D^m(j_\mu) = D^m(\xi_\mu + n_\mu N) = D^m(\xi_\mu)$$

where $0 \leqslant \xi_\mu \leqslant N, \xi_\mu \in Z, n_\mu \in Z$.

Symbolically

$$D^m(j) = D^m(\xi + j^c) = D^m(\xi) \tag{39}$$

where ξ belongs to the fundamental domain and j^c is any vector in the cubic lattice whose components are multiples of N, $j^c = \{n_\mu N\}$. The boundary conditions (39) are invariant under integral Lorentz transformations. From the definition of the group action

$$U_\Lambda f(x) = f(\Lambda^{-1}x)$$

we have for a periodic function, periodic with respect to the space variable

$$f(\xi + j^c) = f(\xi)$$
$$(U_\Lambda f)(\xi + j^c) = f(\Lambda^{-1}\xi + \Lambda^{-1}j^c) = f(\Lambda^{-1}\xi) = (U_\Lambda f)(\xi) \tag{40}$$

where we have used the property

$$\exp\left(i\frac{2\pi}{N}\sum_{\rho,\mu}\Lambda_\mu^\rho n_\rho N m_\mu\right) = 1$$

due to the integral character of the Lorentz transformations.

4. Dirac representation of the Lorentz group and covariant states

Let $L(\alpha)$ be an element of the proper Lorentz group corresponding to the element $\alpha \in SL(2, C)$ and I_s the parity operator. One writes the components of four-momentum as a (2×2) matrix

$$\tilde{p} \equiv p^\mu \sigma_\mu = p^0\sigma_0 + p^i\sigma_i \tag{41}$$

where $\sigma_0 = 1$ and σ_i are the Pauli matrices.

The transformations of $\tilde{p}$ under parity and $SL(2, C)$ are

$$I_s : \tilde{p} \to \tilde{p}^s = p^0\sigma_0 - p^i\sigma_i = (\det \tilde{p})(\tilde{p})^{-1}$$
$$\alpha : \tilde{p} \to \alpha\tilde{p}\alpha^+$$
$$\tilde{p}^s \to (\alpha^+)^{-1}\tilde{p}^s\alpha^{-1}. \tag{42}$$

It follows

$$I_s L(\alpha)I_s^{-1} = L((\alpha^+)^{-1}). \tag{43}$$

The matrix $(\alpha^+)^{-1}$ gives another 2D representation of the Lorentz group non-equivalent to $\alpha \in SL(2, C)$. In order to enlarge the proper Lorentz group by space reflection we take both representations α and $(\alpha^+)^{-1}$.

Let $\pi \equiv \{I, I_s\}$ the space reflection group and $\alpha \in SL(2, C)$, then the semidirect product

$$SL(2, C) \otimes \pi$$

with the multiplication law

$$(\alpha, \pi)(\alpha', \pi') = (\alpha\alpha', \pi\pi') \qquad \text{if } \pi = I \tag{44}$$

$$(\alpha, \pi)(\alpha', \pi') = (\alpha(\alpha'^+)^{-1}, \pi\pi') \qquad \text{if } \pi = I_s \tag{45}$$

form a group.

This group has a four-dimensional (4D) representation, particular elements of which are

$$\overline{D}(\alpha, I) = \begin{pmatrix} \alpha & 0 \\ 0 & (\alpha^+)^{-1} \end{pmatrix} \qquad \overline{D}(e, I_s) = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} \tag{46}$$

that satisfy

$$\overline{D}(e, I_s)\overline{D}(\alpha, I)\overline{D}(e, I_s^{-1}) = \overline{D}((\alpha^+)^{-1}, I). \tag{47}$$

In this representation we could now construct $\overline{D}(S_i)$ for the generators of the Coxeter group. Using formulae (11)–(14) we get the 4D matrices

$$\overline{D}(S_i) = \begin{pmatrix} 0 & \alpha(PS_i) \\ \alpha(PS_i) & 0 \end{pmatrix} \qquad i = 1, 2, 3$$

$$\overline{D}(S_4) = \begin{pmatrix} 0 & [\alpha^+(PS_4)]^{-1} \\ \alpha(PS_4) & 0 \end{pmatrix}$$

$$\overline{D}(S_i)\overline{D}(S_i) = -1 \qquad i = 1, 2, 3, 4$$

$$[\overline{D}(S_1)\overline{D}(S_2)]^3 = \pm 1 \qquad [\overline{D}(S_2)\overline{D}(S_3)]^4 = \pm 1 \qquad [\overline{D}(S_3)\overline{D}(S_4)]^4 = \pm 1.$$

With respect to this representation, a four-component spinor $\overline{\psi}(p)$ in momentum space transforms as follows

$$U(\alpha, I)\overline{\psi}(p) = \overline{D}(\alpha, I)\overline{\psi}(L^{-1}(\alpha)p) \tag{48}$$

$$U(e, I_s)\overline{\psi}(p) = \overline{D}(e, I_s)\overline{\psi}(I_s p). \tag{49}$$

Using a similarity transformation we obtain an equivalent representation

$$D(\alpha, \pi) = M\overline{D}(\alpha, \pi)M^{-1}$$

with

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_0 & \sigma_0 \\ -\sigma_0 & \sigma_0 \end{pmatrix}.$$

In this representation

$$D(\alpha, I) = \frac{1}{2} \begin{pmatrix} \alpha + (\alpha^+)^{-1}, & -\alpha + (\alpha^+)^{-1} \\ -\alpha + (\alpha^+)^{-1}, & \alpha + (\alpha^+)^{-1} \end{pmatrix} \tag{50}$$

$$D(e, I_s) = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}. \tag{51}$$

For this representation we can derive from equation (46) the unitarity relation

$$D^+(\alpha, \pi)D(e, I_s)D(\alpha, \pi) = D(e, I_s).$$

The new four-spinor

$$\psi(p) = M\overline{\psi}(p)$$

transforms as

$$U(\alpha, I)\psi(p) = D(\alpha, I)\psi(L^{-1}(\alpha)p) \tag{52}$$

$$U(e, I_s)\psi(p) = D(e, I_s)\psi(I_s, p) \tag{53}$$

and has an invariant scalar product due to the unitary relation given earlier. We call this representation the Dirac representation.

The Dirac wave equation can be considered as a consequence of the relativistic invariance and irreducibility [10]. Under the restriction to $SU(2)$, the first and second pair of components of the four-spinor transform according to spin $1/2$ representation. Irreducibility requires that one of these pairs should be eliminated. In the present discrete case we must replace the continuous group $SU(2)$ by the binary octahedral group [14]. Fortunately the restriction of $SU(2)$ to this discrete subgroup is irreducible [11]. This allows us to follow the steps of the continuous analysis [10]. In the rest system we want a projection operator that selects one irreducible representation of $SU(2)$ out of the Dirac representation. This is achieved in the rest system by the projection operator

$$Q = \frac{1}{2}(I + \beta) \qquad \beta \equiv \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}. \tag{54}$$

In order to get the projection operator in an arbitrary system we apply $D(\kappa, I)$ given by equations (50) and (16). In the last equation if we identify $M = (P_0/m_0c)$, $R = (P_1/m_0c)$, $S = (P_2/m_0c)$, $T = (P_3/m_0c)$, then

$$Q \rightarrow Q(\kappa) = D^{-1}(\kappa, I)QD(\kappa, I) = \frac{1}{2}(I + W(\kappa)) \tag{55}$$

where

$$W(\kappa) = \frac{1}{2}\begin{pmatrix} \kappa^+\kappa + (\kappa^+\kappa)^{-1}, & -\kappa^+\kappa + (\kappa^+\kappa)^{-1} \\ \kappa^+\kappa - (\kappa^+\kappa)^{-1}, & -\kappa^+\kappa - (\kappa^+\kappa)^{-1} \end{pmatrix}. \tag{56}$$

Using the identities

$$(\kappa^+\kappa)^{-1} = \frac{1}{m_0c}\sigma^\mu p_\mu \qquad \kappa^+\kappa = \frac{1}{m_0c}\sigma^0 p_0 - \sigma^j p_j \tag{57}$$

we find

$$W(\kappa) = \frac{1}{m_0c}\gamma^\mu p_\mu \tag{58}$$

where γ^μ are Dirac matrices with the realization

$$\gamma^0 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix} \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \qquad \gamma^0 = \gamma_0, \quad \gamma^j = -\gamma_j.$$

Collecting these results we obtain the Dirac equation in momentum space

$$Q(\kappa)\psi(p) = \frac{1}{2}(I + W(\kappa))\psi(p) = \psi(p)$$

or

$$(\gamma^\mu p_\mu - m_0cI)\psi(p) = 0. \tag{59}$$

(An equivalent method can be used applying to the projection operator the Foldy–Wouthuysen transformation [9].)

We apply the operator $\gamma_\mu p^\mu + m_0c$ from the left to (59) and obtain the mass-shell condition

$$(p^\mu p_\mu - m_0^2c^2)\psi(p) = 0. \tag{60}$$

The Dirac equation is invariant under the group with elements (α, π) defined before. In other words, if $\psi(p)$ is a solution of the Dirac equation, so is $U(\alpha, \pi)\psi(p)$. Putting $\pi = I$, then

$$U(\alpha, I)\psi(p) = D(\alpha, I)\psi(L^{-1}p)$$
$$D^{-1}(\alpha)Q(\kappa)D(\alpha) = Q(\kappa\alpha)$$
$$Q(\kappa\alpha)\psi(L^{-1}p) = \psi(L^{-1}p).$$

Therefore

$$\begin{aligned}
Q(\kappa)U(\alpha, I)\psi(p) &= Q(\kappa)D(\alpha, I)\psi(L^{-1}(\alpha)p) \\
&= D(\alpha, I)\,Q(\kappa\alpha)\psi(L^{-1}(\alpha)p) \\
&= D(\alpha, I)\psi(L^{-1}(\alpha)p) = U(\alpha, I)\psi(p)
\end{aligned}$$

as required. For the space reflection

$$Q(I_s\kappa)\psi(I_sp) = \psi(I_sp)$$
$$D^{-1}(e, I_s)Q(\kappa)D(e, I_s) = Q(I_sp)$$
$$U(e, I_s)\psi(p) = D(e, I_s)\psi(I_sp)$$

we get

$$\begin{aligned}
Q(\kappa)U(e, I_s)\psi(p) &= Q(\kappa)D(e, I_s)\psi(I_sp) = D(e, I_s)Q(I_s\kappa)\psi(I_sp) \\
&= D(e, I_s)\psi(I_sp) = U(e, I_s)\psi(p)
\end{aligned}$$

as required. Notice that, due to the relation between $SU(2)$ and the binary octahedral group mentioned before, all properties of the Dirac representation in continuous momentum space carry over to the discrete momentum space without modification.

5. Dirac and Klein–Gordon equations on the lattice

From the Dirac equation in momentum space (59) we can construct the wave equation in position space with the help of the Fourier transform we introduced in section 3. We define the following difference operators

$$\Delta f(j) = f(j+1) - f(j) \qquad \tilde{\Delta} f(j) = \tfrac{1}{2}\{f(j+1) + f(j)\} \tag{61}$$

$$\nabla f(j) = f(j) - f(j-1) \qquad \tilde{\nabla} f(j) = \tfrac{1}{2}\{f(j) + f(j-1)\} \tag{62}$$

and the partial difference operators with respect to a function of several discrete variables

$$\Delta_\nu f(j_\mu) = f(j_\mu + \delta_{\mu\nu}) - f(j_\mu)$$
$$\tilde{\Delta}_\nu f(j_\mu) = \tfrac{1}{2}\{f(j_\mu + \delta_{\mu\nu}) + f(j_\mu)\} \tag{63}$$

and similarly $\nabla_\nu f(j_\mu)$ and $\tilde{\nabla}_\nu f(j_\mu)$.

From these operators we construct

$$\delta_\mu^+ \equiv \frac{1}{\varepsilon}\Delta_\mu \prod_{\nu \neq \mu} \tilde{\Delta}_\nu \qquad \delta_\mu^- \equiv \frac{1}{\varepsilon}\nabla_\mu \prod_{\nu \neq \mu} \tilde{\nabla}_\nu \tag{64}$$

$$\eta^+ \equiv \prod_{\mu=0}^{3} \tilde{\Delta}_\mu \qquad \eta^- \equiv \prod_{\mu=0}^{3} \tilde{\nabla}_\mu. \tag{65}$$

From the Fourier transform we can derive the wave equation in lattice space.

5.1. Type I Fourier transform

The kernel of the transform satisfies

$$\frac{1}{\varepsilon}\Delta_\mu \exp(2\pi i(k \cdot j)\varepsilon) = i\frac{2}{\varepsilon}\tan(\pi k_\mu \varepsilon)\tilde{\Delta}_\mu \exp(2\pi i(k \cdot j)\varepsilon). \tag{66}$$

We could apply the Fourier transform to the Dirac equation in momentum space (59) and would obtain the discrete wave equation. Instead we postulate a difference equation that in the limit goes to the continuous differential equation, namely

$$(i\gamma^\mu \delta_\mu^+ - m_0 c\eta^+)\psi(j_\mu) = 0. \tag{67}$$

The kernel of the Fourier transform (25) or 'plane wave' is a particular solution of (67) if it satisfies

$$\left(\gamma^\mu \frac{2}{\varepsilon}\tan \pi k_\mu \varepsilon - m_0 c\right)\eta^+ \exp 2\pi \, i(k \cdot j)\varepsilon = 0. \tag{68}$$

Applying the operator $\gamma^\mu (2/\varepsilon)\tan \pi k_\mu \varepsilon + m_0 c$ from the left side to the last equation we obtain

$$\frac{4}{\varepsilon^2}(\tan \pi k^\mu \varepsilon)(\tan \pi k_\mu \varepsilon) - m_0^2 c^2 = 0 \tag{69}$$

which is the integrability condition for solution of the wave equation.

Now we multiply equation (68) by some arbitrary (periodic) function $\psi(j_\mu)$ of discrete variables and sum for all j_μ.

We get

$$\left(\gamma^\mu \frac{2}{\varepsilon}\tan \pi k_\mu \varepsilon - m_0 c\right)\hat{\psi}(k_\mu) = 0 \tag{70}$$

where

$$\hat{\psi}(k_\mu) = \sum_{j_\mu=0}^{N-1}\psi(j_\mu)\eta^+ \exp(i2\pi k^\mu j_\mu \varepsilon) \tag{71}$$

is the Fourier transform of $\psi(j_\mu)$.

If we compare equation (70) with equation (59) both are identical if we restrict the momentum p_μ to the discrete values

$$p_\mu = \frac{2}{\varepsilon}\tan \pi k_\mu \varepsilon \qquad k_\mu = \frac{1}{2\pi \, i\varepsilon}\ln\frac{1 + (1/2)\,i\varepsilon p_\mu}{1 - (1/2)\,i\varepsilon p_\mu}. \tag{72}$$

Instead of postulating the Dirac equation on the lattice (67) we can deduce it in a natural way from the projection operator of the Dirac representation (59): we first identify the momentum p_μ in this expression with the new variable k_μ as given in (72), then we apply the Fourier transform of type I (23) and finally, using (66), obtain (67).

Applying to the wave equation the operator $i\gamma^\mu \delta_\mu^- + m_0 c\eta^-$ from the left side we obtain the discrete version of the Klein–Gordon equation in the lattice space [6]

$$(\delta_\mu^+ \delta^{\mu-} - m_0^2 c^2 \eta^+ \eta^-)\psi(j_\mu) = 0 \tag{73}$$

a particular solution of which is again the 'plane wave' (25) provided the integrability condition (69) is satisfied.

5.2. Type II Fourier transform

The kernel of this Fourier transform satisfies

$$\frac{1}{\varepsilon}\Delta_\mu \exp\frac{2\pi i}{N}(j\cdot m) = i\frac{2}{\varepsilon}\tan\left(\frac{\pi}{N}m\right)\tilde{\Delta}_\mu \exp\frac{2\pi i}{N}(j\cdot m). \tag{74}$$

We postulate a difference equation on the lattice space the limit of which goes to the continuous wave equation

$$(i\gamma^\mu\delta_\mu^+ - m_0 c\eta^+)\psi(j_\mu) = 0. \tag{75}$$

The kernel of the type II Fourier transforms (38) is a particular solution or 'plane wave' of this equation (75) if it satisfies

$$\left(\gamma^\mu\frac{2}{\varepsilon}\tan\frac{\pi}{N}m_\mu - m_0 c\right)\eta^+ \exp\frac{2\pi i}{N}(j\cdot m) = 0. \tag{76}$$

Applying the operator $\gamma^\mu(2/\varepsilon)\tan(\pi/N)m_\mu + m_0 c$ from the left side we get

$$\frac{4}{\varepsilon^2}\left(\tan\frac{\pi}{N}m_\mu\right)\left(\tan\frac{\pi}{N}m_\mu\right) - m_0^2 c^2 = 0 \tag{77}$$

which is the integrability condition for the 'plane wave solution' of the wave equation on the lattice.

Now we multiply equation (71) by some arbitrary (periodic) function $\psi(j_\mu)$ of discrete variables and sum for all j_μ. We get

$$\left(\gamma^\mu\frac{2}{\varepsilon}\tan\frac{\pi}{N}m_\mu - m_0 c\right)\hat{\psi}(k_\mu) = 0 \tag{78}$$

where

$$\hat{\psi}(k_\mu) = \sum_{j_\mu=0}^{N}\psi(j_\mu)\eta^+ \exp i\frac{2\pi}{N}m^\mu j_\mu \tag{79}$$

is the Fourier transform of $\psi(j_\mu)$.

Both equation (59) and equation (78) are identical if we restrict the momentum p_μ to the discrete values

$$p_\mu = \frac{2}{\varepsilon}\tan\frac{\pi}{N}m_\mu \qquad m_\mu = \frac{N}{2\pi i}\ln\frac{1+(1/2)\,i\varepsilon p_\mu}{1-(1/2)\,i\varepsilon p_\mu} \qquad m_\mu = 0, 1, \ldots, N-1. \tag{80}$$

From equation (75) we could derive the Klein–Gordon equation on the lattice space as in equation (73) with the integrability conditions (69) as before.

As in the case of Fourier transforms of type I we can deduce again the wave equation on the lattice in a natural manner from the projection operator (59): we identify the momentum p_μ with the new variable given by (80), use the Fourier transform (37) of type II, together with (74) and obtain (75).

6. Induced representations of the discrete Poincaré groups

Let $\mathcal{P}_+^\uparrow = T_4 \times_s SO(3,1)$ be the Poincaré group restricted to the integral Lorentz transformations and discrete translations on the lattice with group composition

$$(a, \wedge)(a', \wedge') = (a + \wedge a', \wedge\wedge'). \tag{81}$$

In order to construct irreducible representations we follow the standard method [13].

(1) Choose a unitary irreducible representation (UIR), $D^{\overset{\circ}{k}}(a)$, of the translation group T_4.

(2) Define the little group $H \in SO(3, 1)$ by the stability condition

$$h \in H : D^{\overset{o}{k}}(h^{-1}a) = D^{\overset{o}{k}}(a). \tag{82}$$

(3) Choose a UIR D^α of the little group H and construct for the group $T_4 \times_s H$ the UIR

$$D^{\overset{o}{k},\alpha}(a, h) = D^{\overset{o}{k}}(a) \otimes D^\alpha(h). \tag{83}$$

(4) Choose coset generators c of $T_4 \times_s H$ in $\mathcal{P}_+^\uparrow$ constructed from the group action

$$(\tilde{a}, \tilde{\wedge})c = c'(a, h). \tag{84}$$

(5) It follows that the induced representations is

$$D_{c'c}^{\overset{o}{k},\alpha}(\tilde{a}, \tilde{\wedge}) = D^{\overset{o}{k},\alpha}(a, h)\delta((c')^{-1}(\tilde{a}, \tilde{\wedge})c, (a, h)) \tag{85}$$

and this is a UIR of $\mathcal{P}_+^\uparrow$.

We recall the construction of the massive UIR for the continuous case. If $\overset{o}{k}$ is any vector inside the forward light cone, one can shift it on the orbit by a continuous Lorentz transformation to the form $\overset{o}{k} = m_0 c(1, 0, 0, 0)$, with the stability group $H = SO(3)$. In (2×2) matrix form this stability group becomes $SO(2)$.

For the discrete Poincaré group, we may choose $\overset{o}{k}$ from the intersection of the Brillouin zone with the forward light cone. We can shift it on the orbit only by discrete Lorentz transformations. Certainly we can choose $\overset{o}{k} = m_0 c(1, 0, 0, 0)$ within the Brillouin zone. Then, the discrete stability group is the cubic group and in (2×2) matrix form it becomes the binary cubic group. In both cases the representations subduced from the continuous to the discrete little groups remain irreducible.

For the coset representative $c \equiv (0, \wedge)$ we can choose the integral Lorentz transformations $\wedge \equiv L(k)$ that take $\overset{o}{k}$ into an arbitrary integral vector of the unit hyperboloid. These transformations were defined in section 2 as integral boosts. The Dirac delta function in (85) is zero unless

$$(a, h) = (0, L^{-1}(k'))(\tilde{a}, \tilde{\wedge})(0, L(k)) = (L^{-1}(k')\tilde{a}, L^{-1}(k')\tilde{\wedge}L(k)).$$

Substituting into equation (85) with $c, c' \to K, k'$ and using (82) we get

$$D_{k'k}^{\overset{o}{k},\alpha}(\tilde{a}, \tilde{\wedge}) = D^{\overset{o}{k}}(L^{-1}(k')\tilde{a})D^\alpha(L^{-1}(k')\tilde{\wedge}L(k)). \tag{86}$$

The spinor representation of the second factor is given with respect to the element $L^{-1}(k')\tilde{\wedge}L(k)$ that belongs to the little group, $SU(2)$, or $SO(3, R)$, respectively. These representations of $SU(2)$ corresponding to spin-1/2 or spin-1 state are irreducible when restricted to the (binary) cubic group [11]. The first factor can be written

$$D^{\overset{o}{k}}(L^{-1}(k')\tilde{a}) = D^{k'}(\tilde{a}) \tag{87}$$

where $k' = (L^{-1}(k'))^T \overset{o}{k}$ are all the points that defined the UIR of the translation group and belong to the orbit on the dual translation group. This orbit is discrete in our analysis.

We apply the analysis given in [13] for the semidirect product of the discrete translation group and a point group on the lattice. The dual group of the translation group is given by all the points from the Brillouin zone. We wish to characterize the discrete orbit of the point group by a function on the dual space. We formulate five conditions for these constraints:

(1) they should vanish on the orbit points;
(2) they should admit a periodic extension on the k-space;

(3) the constraints must be Lorentz invariant;

(4) the constraints should vanish only on the points of the orbit;

(5) when the lattice spacing goes to zero, the difference equations in position space should go to the continuous wave equation in Minkowski space.

If we require only conditions (1)–(4) the polynomials $(k^\mu k_\mu - \overset{o}{k}{}^2) = 0$ vanish on, and only on, the discrete points of the orbit and the irreducible representation could be characterized by the functions (see [15, p 192])

$$f^*(k) : (k^\mu k_\mu - \overset{o}{k}{}^2) f^*(k) = 0. \tag{88}$$

Nevertheless the difference equation (73) we constructed in section 5, with the kernel of Fourier transforms of type I and II, does not lead to this constraint. We take a new approach. We choose the constraints we derived in the Dirac representation in momentum space equation (60) for the continuous case

$$(p^\mu p_\mu - m_0^2 c^2) \psi(p) = 0. \tag{89}$$

If we identify $p_\mu = (2/\varepsilon) \tan \pi k_\mu \varepsilon$, with k in the Brillouin zone and ε the lattice spacing and use the Fourier transform of type I we obtain the difference equation (73).

If we identify $p_\mu = (2/\varepsilon) \tan(\pi/N) m_\mu$ in (89) and use the Fourier transform of type II we obtain again the difference equation (73), the continuous limit of which leads to the continuous Klein–Gordon equation.

The constraints (69) are periodic with respect to the k_μ-space in the type I and the constraints (77) are periodic with respect to the m_μ-space in the type II.

Nevertheless when integral Lorentz transformations are applied to the components of the k_μ or m_μ variables the new p_μ do not satisfy the constraint equations and at the same time the constraints vanish at points not on the orbit. Therefore, conditions (3) and (4) are violated, although they can be recovered in the asymptotic limit when $\varepsilon \to 0$

$$(k^\mu k_\mu - \overset{o}{k}{}^2) \psi(k_\mu) = 0 \qquad \text{type I} \tag{90}$$

or in the limit $\varepsilon \to 0$, $N\varepsilon \to 2\pi$

$$(m^\mu m_\mu - 1) \psi(m_\mu) = 0 \qquad \text{type II.} \tag{91}$$

The corresponding representation becomes irreducible and invariant.

In summary, the constraints given by (88) satisfy conditions (1), (2) and (4) but they do not lead to the difference equation (73) via the finite Fourier transform.

The constraints given by (89) satisfy conditions (1) and (2) and they lead to the difference equation (73), although they do not satisfy conditions (3) and (4).

The constraints given by equations (90) and (91) are equivalent to (88) and they lead to the field equation of continuous type, because they are constructed by the limit $\varepsilon \to 0$, which also implies that we recover the continuous wave equation.

In conclusion, the orbit condition for the irreducible representation of the Poincaré group does not satisfy simultaneously all the conditions for the constraints.

This situation is equivalent to the problem of the fermion doubling in lattice gauge theories. Each model does not satisfy all the conditions for the Hamiltonian although they have the advantage of avoiding the infinities appearing in the fields for the continuous case.

Acknowledgments

where part of this work was carried out. This work has been partially supported by DGICYT contract No Pb94-1438 (Spain). We also thank the referees for many illuminating remarks.

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